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Newton's Method and Nonlinear Boundary Value Problems*

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1. INTRODUCTION

Algebraic fixed-point theorems have long been important tools in the investigation of boundary value problems for ordinary differential equations. For example, Picard [6] made extensive use of successive approximation methods during his pioneering studies, and recently Falb and de Jong [2] have used several iterative techniques to investigate two-point nonlinear problems. The standard procedure for applying these constructive methods involves converting the problem to an equivalent integral equation by the choice of a suitable Green's function. The resulting theory is consequently limited to problems for which such a formulation is possible.

In this paper we shall apply a Newton iterative technique to boundary value problems for nonlinear ordinary differential equations. Our approach differs from the traditional work in two ways. First, we treat a very general problem with nonlinear boundary conditions. Moreover, we apply the iterative technique directly to the operators that result from the boundary value problems; no formulation as an integral equation is required. Using this approach, we develop existence and local uniqueness criteria for solutions of both traditional and less familiar boundary value problems.

Let $C(I)$ denote the linear space of continuous functions from the compact interval $I = [a, b]$ into n-dimensional real arithmetic space R_n, and let $C'(I)$ be the subspace of continuously differentiable functions on I. We shall consider the boundary value problem for a first-order system of n ordinary differential equations on I given by

$$x' + F(x, t) = 0, \qquad f(x) = 0. \tag{1.1}$$

The function f is a mapping from a subset of $C'(I)$ into R_m, where m and n

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are not necessarily equal. The problem (1.1) will be referred to as a *nonlinear boundary value problem.*

Very little research has been done in the direction of applying constructive techniques directly to the operators resulting from nonlinear boundary value problems. The previous work is presented in two papers by M. Urabe. In the first [10], he considers multipoint boundary value problems and obtains results for the existence and uniqueness of solutions by applying Banach's contraction mapping principle. In the second paper [11], he applies a similar technique to a problem with more general boundary conditions.

2. Newton's Method in Banach Spaces

In this section, we present a generalization of the well-known theorem of L. V. Kantorovich on the convergence of Newton's method [3]. Although there is a vast literature detailing various extensions of Kantorovich's theorem (see [5] and [7] for bibliographies), none of the extensions is well suited for dealing with the operators resulting from nonlinear boundary value problems. In particular, the available formulations require the existence of a two-sided inverse for a linear operator associated with the problem. Such an assumption would constrain the operator f in (1.1) to be an n-dimensional functional and thus severely limit the class of problems that could be treated. The following generalization of the Kantorovich theorem overcomes this difficulty by requiring only right inverses. Although the theorem and proof embody several variations from those in the literature, sections of the forthcoming development parallel the presentations in [5, pp. 421–424; 7, pp. 135–142; 8].

If X and Y are real Banach spaces, we use $L(X, Y)$ to denote the Banach space of bounded linear operators from X into Y. If Q is an operator from an open subset of X into Y, then $Q'(x)$ and $Q''(x)$ are the first and second Fréchet derivatives of Q at a point x. The open ball with center at x and radius r will be denoted by $S(x, r)$, and its closure will be designated by $\bar{S}(x, r)$. The identity operator on a linear space will be denoted by E.

Theorem 1. *Let X and Y be Banach spaces and suppose D is an open subset of X. Assume $Q\colon D \to Y$ where Q is Fréchet differentiable on D. For $x_0 \in D$ suppose there exists an operator $[Q'(x_0)]^+ \in L(Y, X)$ such that*

$$Q'(x_0)\,[Q'(x_0)]^+ = E.$$

Further assume there exist positive constants β, K, η such that

(i) $\|[Q'(x_0)]^+\| \leqslant \beta$;

(ii) $\|[Q'(x_0)]^+ Q(x_0)\| \leqslant \eta$;

(iii) $\| Q'(x) - Q'(y)\| \leqslant K \| x - y \|$ *for all* $x, y \in \bar{S}(x_0, r_0)$ (2.1) *where* $\bar{S}(x_0, r_0) \subset D$ *and*

$$r_0 = \frac{1 - (1 - 2h)^{1/2}}{\beta K}, \tag{2.2}$$

$$h = \beta K \eta \leqslant \tfrac{1}{2}.$$

Conclusion.

(1) *The Newton iteration for* Q *starting at* x_0, *namely*

$$x_{k+1} = x_k - [Q'(x_k)]^+ Q(x_k), \qquad k = 0, 1, 2, \ldots,$$

yields a sequence $\{x_k\}$ *which remains in* $\bar{S}(x_0, r_0)$ *and converges to* $x^* \in \bar{S}(x_0, r_0)$ *such that* $Q(x^*) = 0$.

(2) *The sequence of right inverse operators generated from* $[Q'(x_0)]^+$ *and given by*

$$[Q'(x_k)]^+, \qquad k = 0, 1, 2, \ldots,$$

is uniquely determined.

(3) *An error estimate is given by*

$$\| x^* - x_k \| \leqslant \frac{1}{\beta K} \frac{(2h)^{2^k}}{2^k}, \qquad k = 0, 1, 2, \ldots.$$

(4a) *If* $h < \frac{1}{2}$ *and* (2.1) *holds in* $S(x_0, r_1)$ *where*

$$r_1 = \frac{1 + (1 - 2h)^{1/2}}{\beta K}, \qquad S(x_0, r_1) \subset D$$

and $[Q'(x_0)]^+$ *is a two-sided inverse of* $Q'(x_0)$, *then* x^* *is the unique solution of* $Q(x) = 0$ *in* $S(x_0, r_1)$.

(4b) *If* $h = \frac{1}{2}$ *and* $[Q'(x_0)]^+$ *is a two-sided inverse of* $Q'(x_0)$, *then* x^* *is the unique solution of* $Q(x) = 0$ *in* $\bar{S}(x_0, r_0)$.

Proof. Let $p(t) = \frac{1}{2}\beta K t^2 - t + \eta$, and define $\{t_k\}$ to be the scalar sequence of Newton iterates starting at $t_0 = 0$ for the equation $p(t) = 0$. The sequence $\{t_k\}$ is strictly increasing and converges to r_0. Furthermore, a calculation shows that for $k \geqslant 1$

$$t_{k+1} - t_k = \frac{\frac{1}{2}h(t_k - t_{k-1})^2}{\eta - h t_k}. \tag{2.3}$$

If $x \in S(x_0, r_0)$, it follows by (2.2) that

$$\| x - x_0 \| < \frac{1}{\beta K}. \tag{2.4}$$

We can write

$$Q'(x) = Q'(x_0)\{E - [Q'(x_0)]^+ (Q'(x_0) - Q'(x))\}$$

and obtain, with the aid of (2.1), (2.4), and hypothesis (i), that

$$\|[Q'(x_0)]^+\| \| Q'(x_0) - Q'(x)\| < 1.$$

Hence by [9, Theor. 4.1-C], we have that the operator

$$M = \{E - [Q'(x_0)]^+ (Q'(x_0) - Q'(x))\}^{-1}$$

exists, belongs to $L(X, X)$, and can be represented as

$$M = E + \sum_{n=1}^{\infty} \{[Q'(x_0)]^+ (Q'(x_0) - Q'(x))\}^n. \tag{2.5}$$

Thus $[Q'(x)]^+$ exists and is given by $[Q'(x)]^+ = M[Q'(x_0)]^+$. Furthermore, the representation (2.5) yields

$$\|[Q'(x)]^+\| \leqslant \frac{\beta}{1 - \beta K \| x - x_0 \|}. \tag{2.6}$$

By hypothesis (ii) it follows that $\| x_1 - x_0 \| \leqslant t_1$. Hence we have that x_1 exists and lies in $S(x_0, r_0)$. Now suppose x_k exists, $x_k \in S(x_0, r_0)$, and $\| x_k - x_{k-1} \| \leqslant t_k - t_{k-1}$ for $k = 1, 2,..., n$. Since Q is Fréchet differentiable and satisfies (2.1) on $S(x_0, r_0)$, we have

$$Q(x_n) = Q(x_{n-1}) + \int_0^1 Q'(\theta x_n + (1 - \theta) x_{n-1}) (x_n - x_{n-1}) \, d\theta.$$

Then by the definition of the Newton sequence, it follows that

$$Q(x_n) = \int_0^1 [Q'(\theta x_n + (1 - \theta) x_{n-1}) - Q'(x_{n-1})] (x_n - x_{n-1}) \, d\theta.$$

Therefore, using (2.1), we conclude

$$\| Q(x_n)\| \leqslant \tfrac{1}{2} K \| x_n - x_{n-1} \|^2. \tag{2.7}$$

Since $[Q'(x_n)]^+$ exists, x_{n+1} is defined as an element of X, and thus by (2.6) and (2.7) we obtain

$$\| x_{n+1} - x_n \| \leqslant \frac{\tfrac{1}{2} h \| x_n - x_{n-1} \|^2}{\eta - h \| x_n - x_0 \|}.$$

Because $\| x_n - x_0 \| \leqslant t_n$, it follows from the induction hypothesis that

$$\| x_{n+1} - x_n \| \leqslant \frac{\frac{1}{2}h(t_n - t_{n-1})^2}{\eta - ht_n}.$$

Using (2.3), we see that $\| x_{n+1} - x_n \| \leqslant t_{n+1} - t_n$. Furthermore, we obtain $\| x_{n+1} - x_0 \| \leqslant t_{n+1}$ and hence $x_{n+1} \in S(x_0, r_0)$. By induction, we have established that, for any positive integer k, x_k exists, $x_k \in S(x_0, r_0)$, and

$$\| x_k - x_{k-1} \| \leqslant t_k - t_{k-1}.$$

It follows immediately that for any positive integers k and q,

$$\| x_{k+q} - x_k \| \leqslant t_{k+q} - t_k \leqslant r_0 - t_k.$$

Therefore $\{x_k\}$ is a Cauchy sequence which converges to some element $x^* \in \bar{S}(x_0, r_0)$ since X is complete. Furthermore the continuity of Q implies that $Q(x_k) \to Q(x^*)$ if $x_k \to x^*$. From the definition of the Newton sequence for Q beginning at x_0, we see that

$$\| Q(x_k) \| = \| Q'(x_k)(x_{k+1} - x_k) \|$$

and thus

$$\| Q(x_k) \| \leqslant (\| Q'(x_0) \| + \| Q'(x_0) - Q'(x_k) \|) \| x_{k+1} - x_k \|.$$

Hence by hypothesis (iii), we obtain

$$\| Q(x_k) \| \leqslant (\| Q'(x_0) \| + Kr_0) \| x_{k+1} - x_k \|.$$

Thus, $Q(x_k) \to 0$ as $k \to \infty$, and by the uniqueness of limits in normed linear spaces, it follows that $Q(x^*) = 0$. This proves (1).

The representation

$$[Q'(x_k)]^+ = \{E - [Q'(x_0)]^+ (Q'(x_0) - Q'(x_k))\}^{-1} [Q'(x_0)]^+$$

is valid at each step of the iteration and shows that the sequence of right inverses is uniquely determined for a fixed choice of $[Q'(x_0)]^+$. This observation establishes (2).

The argument is completed by noting that the error estimate in (3) follows as in [8] and that statements (4a) and (4b) concerning uniqueness can be proved in the same manner as in [7, pp. 139–142].

The following result is more suitable than Theorem 1 for immediate applications to the operators which result from nonlinear boundary value problems.

COROLLARY. *Let X and Y be Banach spaces and D be an open subset of X. Suppose Q is a twice continuously Fréchet differentiable mapping of D into Y. For $x_0 \in D$ assume that the operator $Q'(x_0)$ is right invertible. Furthermore, assume there are positive real numbers α, β, K such that*

(i) $\| Q(x_0) \| \leqslant \alpha$;

(ii) $\| [Q'(x_0)]^+ \| \leqslant \beta$;

(iii) $\| Q''(x) \| \leqslant K$ *for all* $x \in \bar{S}(x_0, r_0)$ (2.8)

where $\bar{S}(x_0, r_0) \subset D$ *and*

$$r_0 = \frac{1 - (1 - 2h)^{1/2}}{\beta K},$$

$$h = \alpha \beta^2 K \leqslant \tfrac{1}{2}.$$

Then the conclusion of Theorem 1 *is valid, provided* (2.1) *is replaced by* (2.8) *in statement* (4a) *about uniqueness.*

Proof. The result follows from the observation that the assumptions of the corollary are stronger than the corresponding hypotheses in Theorem 1.

3. NONLINEAR BOUNDARY VALUE PROBLEMS

Boundary value problem (1.1) will now be formulated as an equivalent operator equation, and the corollary to Theorem 1 will be used to obtain the desired existence and uniqueness criteria for solutions. We begin by making $C(I)$ into a Banach space by giving it the uniform norm $\| \cdot \|_u$ defined by

$$\| x \|_u = \sup_{t \in I} \| x(t) \|, \qquad x \in C(I).$$

The linear space $C'(I)$ will be made into a Banach space with the norm $\| \cdot \|_s$,

$$\| x \|_s = \| x \|_u + \| x' \|_u, \qquad x \in C'(I).$$

We shall also need to consider the product space $Y = C(I) \times R_m$, which is a Banach space under the norm

$$\| [\psi, v] \| = \max\{\| \psi \|_u, \| v \|\}, \qquad [\psi, v] \in Y.$$

It is necessary to require that certain relationships hold among the norms introduced on the various spaces of matrices in the problem. To be specific, let **A**, **B**, and **C** be, respectively, the linear spaces of $j \times k$, $k \times l$, and $j \times l$

real matrices with corresponding norms $\| \cdot \|_1$, $\| \cdot \|_2$, and $\| \cdot \|_3$. Then the norms are said to be ***compatible***, if for all $A \in \mathbf{A}$ and $B \in \mathbf{B}$ we have

$$\| AB \|_3 \leqslant \| A \|_1 \| B \|_2 .$$

If we make the natural identification of elements of R_n with $n \times 1$ matrices, then this notion of compatibility is a generalization of the concept defined in [4, p. 427]. We require that the arithmetic spaces R_n and R_m be given norms which are compatible with the norms introduced on the other spaces of matrices in the forthcoming development.

We shall assume that the function F in (1.1) is at least continuously differentiable on $U \times I$ where U is an open subset of R_n. The domain of the operator f is assumed to be an open subset D of $C'(I)$, and we shall require that $x(t) \in U$, $t \in I$, for every choice of $x \in D$. Hence, we can define an operator T on D by

$$T(x)\,(t) = F(x(t), t), \qquad a \leqslant t \leqslant b.$$

Since F is continuously differentiable, it follows that T maps D into $C(I)$ and that T is continuously Fréchet differentiable on D. For any $x_0 \in D$, the value of $T'(x_0)$ at $x \in C'(I)$ can be represented as

$$(T'(x_0)\, x)\,(t) = G(t)\, x(t), \qquad a \leqslant t \leqslant b,$$

where G is an $n \times n$ matrix of continuous functions and the indicated multiplication is ordinary multiplication of a matrix by a vector (see [7, p. 95]). In the remainder of this exposition we shall identify $T'(x_0)$ with the matrix G by using the notation $T'(x_0)\,(t) = G(t)$, $t \in I$.

Now consider the problem (1.1). Define the operator $P\colon D \to Y$ by

$$P(x) = [x' + T(x), f(x)]. \tag{3.1}$$

The following key result is an immediate consequence of the form of P and the definition of the operations on Y.

THEOREM 2. *A function $x \in C'(I)$ is a solution of* (1.1) *if and only if it is a solution of* $P(x) = 0$.

We can now study questions of existence and uniqueness for solutions of (1.1) by applying the Newton iterative technique of the previous section to the equation $P(x) = 0$.

THEOREM 3. *Suppose the operators T and f in* (3.1) *are twice continuously Fréchet differentiable on D. For $x_0 \in D$ let Φ be a fundamental matrix on I for*

$$x' + T'(x_0)\, x = 0$$

and define

$$K_1 = \sup_{t \in I} \| \Phi(t) \| , \qquad K_2 = \sup_{t \in I} \| \Phi^{-1}(t) \| .$$

Also define the linear operator L on R_n by $L\xi = f'(x_0)(\Phi\xi)$, $\xi \in R_n$, and let M be an $m \times n$ matrix representation of L. Assume there exist positive constants α, β, K and an $n \times m$ matrix M^+ such that

(i) $\| P(x_0) \| \leqslant \alpha$;

(ii) $MM^+ = E_m$ *(the $m \times m$ identity matrix)*;

(iii) $1 + (1 + \sup_{t \in I} \| T'(x_0)(t) \|) K_1 \{ \| M^+ \| + (1 + \sup_{t \in I} \| T'(x_0)(t) \|) \times K_1 K_2 \| M^+ \| \| f'(x_0) \| (b - a) + \| M^+ \| \| f'(x_0) \| + K_2(b - 1) \} \leqslant \beta$;

(iv) $\| P''(x) \| \leqslant K$ *for all* $x \in \bar{S}(x_0, r_0)$ (3.2)

where $\bar{S}(x_0, r_0) \subset D$ *and*

$$P''(x) z_1 z_2 = [T''(x) z_1 z_2 , f''(x) z_1 z_2], \qquad z_1 , z_2 \in C'(I), \tag{3.3}$$

$$r_0 = \frac{1 - (1 - 2h)^{1/2}}{\beta K},$$

$$h = \alpha \beta^2 K \leqslant \tfrac{1}{2}.$$

Conclusion. The Newton iteration for P starting at x_0, namely

$$x_{k+1} = x_k - [P'(x_k)]^+ P(x_k), \qquad k = 0, 1, 2,...,$$

yields a sequence $\{x_k\}$ which is contained in $\bar{S}(x_0, r_0)$ and converges to an element $x^* \in \bar{S}(x_0, r_0)$ such that $P(x^*) = 0$. For any nonnegative integer k, an error estimate is given by

$$\| x^* - x_k \|_s \leqslant \frac{1}{\beta K} \frac{(2h)^{2^k}}{2^k}.$$

Furthermore, if $h < \frac{1}{2}$ and (3.2) holds in the open ball $S(x_0, r_1)$ where

$$r_1 = \frac{1 + (1 - 2h)^{1/2}}{\beta K}, \qquad S(x_0, r_1) \subset D$$

and M^+ is an $n \times n$ invertible matrix, then x^* is the unique solution of $P(x) = 0$ in $S(x_0, r_1)$. If $h = \frac{1}{2}$, then x^* is the unique solution of $P(x) = 0$ in $\bar{S}(x_0, r_0)$ provided M^+ is an $n \times n$ invertible matrix.

Proof. We proceed by satisfying the hypotheses of the corollary to Theorem 1. By our choice of norms on $C(I)$ and $C'(I)$ we have that the mapping $H_0 : x \to x'$ is Fréchet differentiable on D. Hence, for $x_0 \in D$,

$$P'(x_0) x = [x' + T'(x_0) x, f'(x_0) x], \qquad x \in C'(I).$$

Thus, we can deal with the problem of calculating and bounding $[P'(x_0)]^+$ by considering linear boundary value problems of the form

$$x' + T'(x_0)\,x = \psi, \tag{3.4}$$

$$f'(x_0)\,x = v \tag{3.5}$$

for $[\psi, v] \in Y$. If we let $Hx = x' + T'(x_0)\,x$, it follows that the inverse image of any $\psi \in C(I)$ under H is the set of solutions of (3.4) and is represented by the following linear variety in $C'(I)$:

$$\left\{\int_a^t \Phi(t)\,\Phi^{-1}(s)\,\psi(s)\,ds\right\} + \mathbf{N}(H)$$

where $\mathbf{N}(H)$ denotes the null space of H. Thus H maps $C'(I)$ onto $C(I)$ and Eq. (3.4) has solutions for every choice of $\psi \in C(I)$. Furthermore, the null space of H is isomorphic to R_n under the isomorphism defined by

$$\xi \leftrightarrow \Phi\xi, \qquad \xi \in R_n\,. \tag{3.6}$$

Because the operator H is onto, it has right inverses, and one such right inverse is given by

$$(H^+\psi)\,(t) = \int_a^t \Phi(t)\,\Phi^{-1}(s)\,\psi(s)\,ds, \qquad a \leqslant t \leqslant b.$$

Hence, using (3.6), we see that all solutions of (3.4) can be represented by

$$x = \Phi\xi + H^+\psi, \qquad \xi \in R_n\,. \tag{3.7}$$

Therefore, x will be a solution of (3.4) and (3.5) if $\xi \in R_n$ is a solution of

$$M\xi = v - f'(x_0)\,H^+\psi. \tag{3.8}$$

By assumption, M has a right inverse M^+, and thus,

$$\xi = \xi_0 + M^+(v - f'(x_0)\,H^+\psi)$$

is a solution of (3.8) for each $\xi_0 \in \mathbf{N}(L)$. However, we must choose $\xi_0 = 0$ in order to obtain a linear right inverse operator, and hence by (3.7) we have

$$[P'(x_0)]^+\,[\psi, v] = \Phi(M^+v - M^+f'(x_0)\,H^+\psi) + H^+\psi. \tag{3.9}$$

Now consider the problem of estimating the norm of $[P'(x_0)]^+ \in L(Y, C'(I))$. For any $[\psi, v] \in Y$, we let $[P'(x_0)]^+\,[\psi, v] = z$. Then by definition,

$$\|[P'(x_0)]^+\| = \sup_{[\psi,v]\neq[0,0]} \frac{\|z\|_u + \|z'\|_u}{\max\{\|\psi\|_u\,, \|v\|\}}\,.$$

But $z' = -T'(x_0)\, z + \psi$ and thus,

$$\|[P'(x_0)]^+\| = \sup_{[\psi, v] \neq [0,0]} \frac{\| z \|_u + \| -T'(x_0)\, z + \psi \|_u}{\max\{\| \psi \|_u , \| v \|\}} . \tag{3.10}$$

Using (3.9) and (3.10), we can apply the boundedness of the linear operator $f'(x_0)$, the compatibility of the norms, and hypothesis (iii) to show that $\|[P'(x_0)]^+\| \leqslant \beta$.

Since the mapping H_0 is a bounded linear operator from $C'(I)$ to $C(I)$, it follows that $H_0''(x) = 0$ for every $x \in C'(I)$. Thus, because T and f are twice Fréchet differentiable on D, so is P and for each $x \in D$, we obtain (3.3). Hence, if hypothesis (iv) holds, a check of the assumptions shows that the conditions of the corollary are satisfied. Our conclusion then follows immediately.

4. Specific Boundary Conditions

The boundary value problem treated in Theorem 3 is very extensive in that it includes nonlinear differential equations subject to many different classes of boundary conditions. The existence and uniqueness criteria developed for it are correspondingly general in nature. In particular, the estimate for the norm of $\|[P'(x_0)]^+\|$ given in hypothesis (iii) is very crude. For specific classes of problems and particular examples, more precise conditions can be formulated. As an illustration, we conclude by treating a problem that has been touched upon in its linear form (see [1, pp. 148–149]) but for which no nonlinear theory exists. The general framework for the following theorem will be the same as in the previous section; in particular, the linear spaces will have the same norms and the operator T is assumed to be derived from a continuously differentiable function F. Unless otherwise specified, all summations are assumed to be taken from 1 to infinity.

Theorem 4. *Suppose $P\colon D \to Y$ where D is an open subset of $C'(I)$ and P has the form*

$$P(x) = \left[x' + T(x), \sum B_i x(\tau_i) - c\right]$$

with $c \in R_m$, $\{B_i\}$ a sequence of real $m \times n$ matrices such that $\sum \| B_i \| < \infty$, and $\tau_i \in I$ for $i = 1, 2,\ldots$. Assume the operator T is twice continuously Fréchet differentiable on D. For $x_0 \in D$ let Φ be a fundamental matrix on I for

$$x' + T'(x_0)\, x = 0$$

and define

$$K_1 = \sup_{t\in I} \| \Phi(t)\| , \qquad K_2 = \sup_{t\in I} \| \Phi^{-1}(t)\| .$$

Assume there exist positive constants α, β, K *and an* $n \times m$ *matrix* M^+ *such that*

(i) $\| P(x_0)\| \leqslant \alpha$;

(ii) $MM^+ = E_m$, *where* $M = \sum B_i\Phi(\tau_i)$;

(iii) $$1 + (1 + \sup_{t\in I} \| T'(x_0)(t)\|) \times K_1 \left\{\| M^+ \| + K_1K_2 \| M^+ \| (b-a) \sum \| B_i \| + K_2(b-a)\right\} \leqslant \beta;$$

(iv) $\| P''(x)\| \leqslant K$ *for all* $x \in \bar{S}(x_0, r_0)$

where $\bar{S}(x_0, r_0)$ CD *and*

$$P''(x)\, z_1z_2 = [T''(x)\, z_1z_2 , 0], \qquad z_1, z_2 \in C'(I),$$
$$r_0 = \frac{1-(1-2h)^{1/2}}{\beta K},$$
$$h = \alpha\beta^2K \leqslant \tfrac{1}{2}.$$

Then the conclusion of Theorem 3 *holds.*

Proof. Let $f(x) = \sum B_ix(\tau_i) - c$. The operator $f_1(x) = \sum B_ix(\tau_i)$ is clearly linear on $C'(I)$. For every positive integer p, we have

$$\left\| \sum_{i=1}^{p} B_ix(\tau_i) \right\| \leqslant \left(\sum \| B_i \|\right) \| x \|_s ,$$

and so f_1 is bounded with a norm no larger than $\sum \| B_i \|$. Thus, P is twice continuously Fréchet differentiable on D. We also note that the finiteness of $\sum \| B_i \|$ guarantees the convergence of $\sum B_i$ and $\sum B_i\Phi(\tau_i)$ and the convergence of $\sum B_ix(\tau_i)$ for any $x \in C'(I)$.

We have for every $x \in C'(I)$ that

$$P'(x_0)\, x = \left[x' + T'(x_0)\, x, \sum B_ix(\tau_i)\right] .$$

Therefore, we consider the boundary value problem

$$x' + T'(x_0)\, x = \psi, \qquad \sum B_ix(\tau_i) = v,$$

for $[\psi, v] \in Y$. Since the matrix $M = \sum B_i\Phi(\tau_i)$ is a representation of the linear operator $\xi \to f'(x_0)(\Phi\xi)$, $\xi \in R_n$, the assumed right invertibility of M

guarantees the existence of a right inverse for $P'(x_0)$. Using the boundedness of $f'(x_0)$ and the compatibility of the norms, we obtain from hypothesis (iii) that $\|[P'(x_0)]^+\| \leqslant \beta$. Furthermore, the linearity of f_1 assures that on D, $P''(x) = [T''(x), 0]$. Hence, the corollary to Theorem 1 applies and the conclusion follows directly.

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